Cubic Dirac operator for $U_q(\mathfrak{sl}_2)$

Andrey Krutov

Institute of Mathematics, Czech Academy of Sciences

Joint with Pavle Pandžić

arXiv:2209.09591 [math.RT]

Representation Theory XVII
October 4, 2022
Dubrovnik

The *noncommutative Weil algebra* of $\mathfrak{g}$

$$\mathcal{W}(\mathfrak{g}) := U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g}).$$

Let e_a denotes the basis of $\mathfrak{g}$ and f_a be the corresponding dual basis. The elements $u_a = e_a \otimes 1$ and $x_a = 1 \otimes f_a$ are generators of $\mathcal{W}(\mathfrak{g})$. Set

$$D := \sum_a u_a x_a + \gamma, \qquad \gamma \in \mathrm{Cl}^{(3)}(\mathfrak{g})$$

The element D may be viewed as a cubic Dirac operator The square D^2 is given by

$$D^2 = \mathrm{Cas}_{\mathfrak{g}} + \frac{1}{24}\,\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}}),$$

where $\mathrm{Cas}_{\mathfrak{g}} = \sum_a e_a f_a$ is the Casimir element of $U(\mathfrak{g})$ and $\mathrm{tr}(\mathrm{Cas}_{\mathfrak{g}})$ is its trace in the adjoint representation of $\mathfrak{g}$.

• Dirac cohomology and Vogan's conjecture (proved by Huang and Pandžić)

• Cartan's model and equivariant cohomologies (Alekseev and Meinrenken)

• Multiples of representation and an algebraic version of Borel–Weil theorem (Kostant).

• Previous works of Kulish, Đurđević, D'Andrea, Dabrowski, Krahmer, Tucker-Simmons, Matassa, Ó Buachalla, Somberg, Das, . . . (geometric setting).

• Gauge theory on noncommutative principal bundles (Ćaćić, Mesland)

• Previous works of Pandžić and Somberg (algebraic setting).

An associative algebra over $\mathbb{K}$ is a 3-tuple (A, m, η)

$$m\colon A \otimes A \to A, \qquad \eta\colon \mathbb{C} \to A.$$

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes m"] \arrow[d, "m \otimes \mathrm{id}"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} \otimes A \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\cong"] & A \otimes A \arrow[d, "m"] & A \otimes \mathbb{K} \arrow[l, "\mathrm{id} \otimes \eta"'] \arrow[dl, "\cong"'] \\
& A &
\end{tikzcd}$$

A coassociative coalgebra over $\mathbb{K}$ is a 3-tuple (A, Δ, ε)

$$\Delta\colon A \to A \otimes A, \qquad \varepsilon\colon A \to \mathbb{C}.$$

$$\begin{tikzcd}
A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta \otimes \mathrm{id}"'] \\
A \otimes A \arrow[u, "\mathrm{id} \otimes \Delta"'] & A \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{K} \otimes A & A \otimes A \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \arrow[r, "\mathrm{id} \otimes \varepsilon"] & A \otimes \mathbb{K} \\
& A \arrow[ul, "\cong"] \arrow[u, "\Delta"'] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

A Hopf algebra over $\mathbb{K}$ is a 6-tuple $(A, m, \eta, \Delta, \varepsilon, S)$, $S\colon A \to A$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

Example

Let G be a finite group, $A = \mathbb{K}G$. For $g \in G$, we have

$$\Delta g = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1}.$$

The tensor algebra $T(V)$ of V. For $v \in V$,

$$\Delta v = v \otimes 1 + 1 \otimes v, \qquad \varepsilon(v) = 0, \qquad S(v) = -v.$$

The universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. For $x \in \mathfrak{g}$,

$$\Delta x = x \otimes 1 + 1 \otimes x, \qquad \varepsilon(x) = 0, \qquad S(x) = -x.$$

If V and W are $\mathfrak{g}$-modules then $\Delta x \in \mathfrak{g} \otimes \mathfrak{g}$ defines the action of x on $V \otimes W$.
The counit $\varepsilon \colon U(\mathfrak{g}) \to \mathbb{C}$ define the trivial representation.

Sweedler notation

$$\Delta \colon H \to H \otimes H, \quad \Delta h = \sum_i x_i \otimes y_i = h_{(1)} \otimes h_{(2)}$$

Drinfel'd–Jimbo Quantum Groups: $\mathfrak{sl}_2$ case

Fix $q \in \mathbb{C}$ such that q is not a root of unity. The *quantised universal enveloping algebra of* $\mathfrak{sl}_2$ is the algebra with four generators E, F, K, K^{-1} satisfying the defining relations

$$KK^{-1} = K^{-1}K = 1, \qquad KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F,$$
$$[E, F] = EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The Hopf algebra structure is given by

$$\Delta(E) = E \otimes K + 1 \otimes E,\ \Delta(F) = F \otimes 1 + K^{-1} \otimes F,\ \Delta(K) = K \otimes K,$$
$$S(K^{\pm 1}) = K^{\mp 1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF,$$
$$\varepsilon(K^{\pm 1}) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0.$$

- $U(\mathfrak{sl}_2)$ is generated by E, F, H.
- Formally set $q = e^{\hbar}$, $K = e^{\hbar}$ in $U_q(\mathfrak{sl}_2)$ and $\hbar \to 0$.
- Let $\widetilde{U}_q(\mathfrak{sl}_2)$ be an algebra generated by E, F, K, K^{-1} and G satisfying

$$[G, E] = E(qK + q^{-1}K^{-1}), \quad [G, F] = -(qK + q^{-1}K^{-1})F,$$
$$[E, F] = G, \quad (q - q^{-1})G = K - K^{-1}.$$

- If $q^2 \neq 1$ then $\widetilde{U}_q(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$ are isomorphic

$$E \mapsto E, \quad F \mapsto F, \quad G \mapsto (q - q^{-1})^{-1}(K - K^{-1})$$

$U(\mathfrak{sl}_2)$ and $\widetilde{U}_1(\mathfrak{sl}_2)$ are closely related. Indeed

$$\widetilde{U}_1(\mathfrak{sl}_2) \simeq U(\mathfrak{sl}_2) \otimes \mathbb{C}\mathbb{Z}_2, \quad U(\mathfrak{sl}_2) \simeq \widetilde{U}_1(\mathfrak{sl}_2)/\langle K-1\rangle$$

For $q = 1$ we have that K belongs to the centre of $\widetilde{U}_1(\mathfrak{sl}_2)$ and the first isomorphism is given by

$$E \mapsto E\mathcal{X}, \quad F \mapsto F, \quad G \mapsto H\mathcal{X},$$

where $\mathcal{X}$ is the generator of $\mathbb{C}\mathbb{Z}_2$ such that $\mathcal{X}^2 = 1$.
Remark. Twice more representations due to $\mathbb{C}\mathbb{Z}_2$.

Let α be a simple root of $\mathfrak{sl}_2$ and λ be an integral weight of $\mathfrak{sl}_2$.
• the Verma module M_λ over $U_q(\mathfrak{sl}_2)$ generated by v_λ with relations

$$Ev_\lambda = 0 \qquad Kv_\lambda = q^{(\lambda,\alpha^\vee)}v_\lambda$$

where $\alpha^\vee$ is the corresponding simple coroot.
• If $\mathfrak{sl}_2$ is a dominant weight of $\mathfrak{g}$ then M_λ has a maximal proper submodule I_λ generated by $F^{(\lambda,\alpha^\vee)+1}v_\lambda$ and

$$V_\lambda := M_\lambda/I_\lambda$$

is a finite-dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$.
• Such representations are called *type-1 representations*.

The left adjoin action of $U_q(\mathfrak{sl}_2)$ on itself is defined by

$$\mathrm{ad}_a b = a_{(1)} b S(a_{(2)}) \qquad \text{for } a, b \in U_q(\mathfrak{sl}_2).$$

In particular, for $b \in U_q(\mathfrak{sl}_2)$,

$$\mathrm{ad}_E b = EbK^{-1} - bEK^{-1}, \qquad \mathrm{ad}_F b = Fb - K^{-1}bKF,$$
$$\mathrm{ad}_K b = KbK^{-1}, \qquad \mathrm{ad}_{K^{-1}} b = K^{-1}bK.$$

Denote

$$v_2 = E,$$
$$v_0 = q^{-2}EF - FE = (q - q^{-1})^{-1}(K - K^{-1}) - q^{-1}(q - q^{-1})EF,$$
$$v_{-2} = KF.$$

Let $\pi \in \mathcal{P}$ be the fundamental weight of $\mathfrak{sl}_2$. The elements v_2, v_0, v_{-2} spans $U_q(\mathfrak{sl}_2)$-module $V_{2\pi}$ with respect to the left adjoint action.

$$\mathrm{ad}_E v_2 = 0, \qquad \mathrm{ad}_K v_2 = q^2 v_2, \qquad \mathrm{ad}_F v_2 = -v_0,$$
$$\mathrm{ad}_E v_0 = -(q + q^{-1})v_2, \qquad \mathrm{ad}_K v_0 = v_0, \qquad \mathrm{ad}_F v_0 = (q + q^{-1})v_{-2}$$
$$\mathrm{ad}_E v_{-2} = v_0, \qquad \mathrm{ad}_K v_{-2} = q^{-2}v_{-2}, \qquad \mathrm{ad}_F v_{-2} = 0.$$

Let C be a monoidal category with the collection of associativity constrains

$$\alpha_{A,B,C} \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C) \qquad A, B, C \in \mathsf{Obj}(\mathsf{C}).$$

A *braiding* on a monoidal category C is a natural isomophism σ between functors $- \otimes -$ and $- \otimes^{\mathsf{op}} -$ such that the hexagonal diagrams commute,

$$\begin{tikzcd}
 & A \otimes (B \otimes C) \arrow[r, "\sigma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{B,C,A}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{A,B,C}"] \arrow[dr, "\sigma_{A,B} \otimes \mathrm{id}_C"'] & & & B \otimes (C \otimes A) \\
 & (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[ur, "\mathrm{id}_B \otimes \sigma_{A,C}"'] &
\end{tikzcd}$$

A *braided* monoidal category is a pair consisting of a monoidal category and a braiding.

If C is a strict braided monoidal category with braiding σ then for all $A, B, C \in$ Obj(C) the braiding satisfies the following Yang–Baxter equation

$$
\begin{tikzcd}
& B \otimes A \otimes C \arrow[rr, "\mathrm{id}_B \otimes \sigma_{A,C}"] & & B \otimes C \otimes A \arrow[dr, "\sigma_{B,C} \otimes \mathrm{id}_A"] & \\
A \otimes B \otimes C \arrow[ur, "\sigma_{A,B} \otimes \mathrm{id}_C"] \arrow[dr, "\mathrm{id}_A \otimes \sigma_{B,C}"] & & & & C \otimes B \otimes A \\
& A \otimes C \otimes B \arrow[rr, "\sigma_{A,C} \otimes \mathrm{id}_C"] & & C \otimes A \otimes B \arrow[ur, "\mathrm{id}_C \otimes \sigma_{A,B}"] &
\end{tikzcd}
$$

A *symmetric* monoidal category is a braided monoidal category such that $\sigma^2 = \mathrm{id}$.

Example

$\mathsf{Vect}_{\mathbb{K}}$, $\sigma(v \otimes w) = w \otimes v$.
Note that

$$S^2 V = \{v \in \mathcal{T}(V) \mid \sigma(v) = v\}, \qquad \Lambda^2 V = \{v \in \mathcal{T}(V) \mid \sigma(v) = -v\}.$$

$$\Lambda V = \mathcal{T}(V)/\langle S^2 V\rangle, \qquad SV = \mathcal{T}(V)/\langle \Lambda^2 V\rangle.$$

Example

$\mathsf{SVect}_{\mathbb{K}}$, $\sigma(v \otimes w) = (-1)^{p(w)p(v)} w \otimes v$.

- $\mathsf{Rep}_1 U_q(\mathfrak{g})$ is a braided monoidal category
- the universal R-matrix $R \in U_q(\mathfrak{g})\widehat{\otimes}U_q(\mathfrak{g})$

$$\rho_V : U_q(\mathfrak{g}) \to \mathrm{End}(V), \qquad \rho_W : U_q(\mathfrak{g}) \to \mathrm{End}(W)$$

$$\sigma_{R,V\otimes W} := \tau \circ (\rho_V \otimes \rho_W)(R), \tag{1}$$

Eigenvalues: $\pm q^{(\cdots)}$ on $V \otimes V$

- the *normalised braiding*

$$\tilde{\sigma}_{R,V\otimes W} := \sqrt{\sigma^{-1}_{R,W\otimes V}\sigma^{-1}_{R,V\otimes W}}\,\sigma_{R,V\otimes W}.$$

Eigenvalues: ± 1 on $V \otimes V$

- $\tilde{\sigma}_{R,V\otimes W}$ does not satisfies the Yang–Baxter equation.
- For any $V \in \mathsf{Rep}_1(U_q(\mathfrak{g}))$, let us denote

$$S^2_q V := \{x \in V\otimes V \mid \tilde{\sigma}_R(x) = x\}, \quad \Lambda^2_q V := \{x \in V\otimes V \mid \tilde{\sigma}_R(x) = -x\}.$$

- the *BZ quantum exterior algebra* $\Lambda_q(V)$ of V to be

$$\Lambda_q(V) := \mathcal{T}(V)/\langle S^2_q V\rangle,$$

For $U_q(\mathfrak{sl}_2)$,

$$R_0 = q^{\hbar H \otimes H/2}, \qquad R_1 = \sum_{m=0}^{+\infty} \frac{q^{m^2-m}(q-q^{-1})^m}{[m]_{q^2}!} E^m \otimes F^m,$$

where $K = q^{\hbar H}$,

$$[m]_{q^2} = \frac{q^{2m}-1}{q^2-1}, \qquad [m]_{q^2}! = [m]_{q^2}[m-1]_{q^2} \dots [1]_{q^2}.$$

The corresponding braiding σ_R on $\mathsf{Rep}_1 U_q(\mathfrak{sl}_2)$ is given by

$$\sigma_R := \tau \circ R \colon V \otimes W \to W \otimes W, \quad R_0(v \otimes w) = q^{(\mathrm{wt}(v), \mathrm{wt}(w))} v \otimes w,$$

where W and V are objects in $\mathsf{Rep}_1 U_q(\mathfrak{sl}_2)$ and $v \in V$, $w \in W$.

For $U_q(\mathfrak{sl}_2)$, the algebra $\Lambda_q V_{2\pi}$ has the classical dimension.

$$\begin{aligned} v_2 \wedge v_2 &= 0, & v_{-2} \wedge v_{-2} &= 0, \\ v_0 \wedge v_2 &= -q^{-2} v_2 \wedge v_0, & v_{-2} \wedge v_0 &= -q^{-2} v_0 \wedge v_{-2}, \\ v_0 \wedge v_0 &= \frac{(1-q^4)}{q^3} v_2 \wedge v_{-2}, & v_{-2} \wedge v_2 &= -v_2 \wedge v_{-2}. \end{aligned}$$

Let A be a Hopf algebra and V be an A-module. A bilinear form $\langle \cdot, \cdot \rangle$ on V is invariant if

$$\langle a_{(1)} v, a_{(2)} w \rangle = \varepsilon(a) \langle v, w \rangle \quad \text{for all } a \in A,\ v, w \in V.$$

The $U_q(\mathfrak{sl}_2)$-module $V_{2\pi}$ admits a nondegenerate invariant bilinear form given by

$$\langle v_2, v_{-2} \rangle = c, \quad \langle v_0, v_0 \rangle = q^{-3}(1+q^2)c, \quad \langle v_{-2}, v_2 \rangle = cq^{-2},$$

where $c \in \mathbb{C}[q, q^{-1}]$.
Note that $\langle \cdot, \cdot \rangle$ is invariant with respect to σ.

Definition

Let $\mathrm{Cl}_q(V_{2\pi}, \sigma, \langle\cdot,\cdot\rangle) := T(V_{2\pi})/I$, where the corresponding two-sided ideal I is generated by

$$x \otimes y + \sigma(x \otimes y) - 2\langle x, y\rangle 1 \qquad \text{for all } x, y \in V_{2\pi}, \tag{2}$$

and σ is the normalized braiding for $V_{2\pi} \otimes V_{2\pi}$.

In what follows we refer to $\mathrm{Cl}_q(V_{2\pi}, \sigma, \langle\cdot,\cdot\rangle)$ as the q-deformed Clifford algebra of $\mathfrak{sl}_2$ and denote it by $\mathrm{Cl}_q(\mathfrak{sl}_2)$. Note that the algebra $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is an associative algebra in the braided monoidal category of $U_q(\mathfrak{sl}_2)$-module, since the ideal (2) is invariant under the action of $U_q(\mathfrak{sl}_2)$.

The generators of the ideal (2) are

$$v_2 \otimes v_2,$$
$$v_0 \otimes v_2 + q^{-2} v_2 \otimes v_0,$$
$$v_{-2} \otimes v_2 - q^{-1} v_0 \otimes v_0 + q^{-4} v_2 \otimes v_{-2},$$
$$q^2 v_{-2} \otimes v_0 + v_0 \otimes v_{-2},$$
$$v_{-2} \otimes v_{-2},$$
$$\tfrac{2(q^2+1)}{q^3} v_{-2} \otimes v_2 + 2v_0 \otimes v_0 + \tfrac{2(q^2+1)}{q} v_2 \otimes v_{-2} - \tfrac{2(q^2+1)(q^4+q^2+1)}{q^5} c1,$$

where $c \in \mathbb{C}[q, q^{-1}]$.
Note that since the ideal generated by (2) is homogeneous with respect to the standard $\mathbb{Z}_2$-grading in the tensor algebra $T(V_{2\pi})$, the algebra $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is also $\mathbb{Z}_2$-graded.

Lemma

The algebra $\mathrm{Cl}_q(\mathfrak{sl}_2)$ *is of the PBW type.*

Proof.

Consider the corresponding homogeneous quadratic algebra $\Lambda_q V_{2\pi}$. Since the Hilbert–Poincare series of $\Lambda_q V_{2\pi}$ is the same in the classical case then $\Lambda_q V_{2\pi}$ is a Koszul algebra. Hence, the algebra $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is of the PBW type. $\square$

$$\begin{aligned}
v_2 v_2 &= 0, & v_{-2} v_{-2} &= 0, \\
v_0 v_2 &= -q^{-2} v_2 v_0, & v_{-2} v_0 &= -q^{-2} v_0 v_{-2}, \\
v_0 v_0 &= \frac{(1-q^4)}{q^3} v_2 v_{-2} + \frac{q^2+1}{q} c1, & v_{-2} v_2 &= -v_2 v_{-2} + \frac{q^2+1}{q^2} c1,
\end{aligned}$$

where $c \in \mathbb{C}[q, q^{-1}]$.

The first remark is that there is a non-scalar central element

$$\gamma = v_2 v_0 v_{-2} + c v_0.$$

The square of γ is computed to be a scalar, $c^2 t^2$, where

$$t = c\sqrt{\frac{q^2+1}{q}}.$$

This now implies there are two orthogonal central projectors in our algebra, one proportional to $\gamma_1 = \gamma - ct$, and the other to $\gamma_2 = \gamma + ct$. It is now easy to check that our algebra is the direct sum of the two ideals I_1, I_2 generated by γ_1 and γ_2.

Let S_1 be a two-dimensional vector space. We consider the representation of $\mathrm{Cl}_q(\mathfrak{sl}_2)$ on S_1 given by

$$v_2 \text{ acts by } \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \qquad v_0 \text{ acts by } \begin{pmatrix} t/q^2 & 0 \\ 0 & -t \end{pmatrix},$$
$$v_{-2} \text{ acts by } \begin{pmatrix} 0 & 0 \\ t/q & 0 \end{pmatrix}.$$

It is easily computed that γ acts by the scalar $-ct$. Moreover, it is clear that our algebra maps onto $\mathrm{End}(S_1)$, so since the ideal I_1 acts by 0, the ideal I_2 is isomorphic to $\mathrm{End}(S_1)$.

Let S_2 be a two-dimensional vector space. We consider the representation of $\mathrm{Cl}_q(\mathfrak{sl}_2)$ on S_2 given by

$$v_2 \text{ acts by } \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \qquad v_0 \text{ acts by } \begin{pmatrix} -t/q^2 & 0 \\ 0 & t \end{pmatrix},$$
$$v_{-2} \text{ acts by } \begin{pmatrix} 0 & 0 \\ t/q & 0 \end{pmatrix}.$$

Now γ acts by the scalar ct. Therefore, S_1 and S_2 are not isomorphic as $\mathrm{Cl}_q(\mathfrak{sl}_2)$-modules. The algebra maps onto $\mathrm{End}(S_2)$, I_2 acts by 0, and I_1 is isomorphic to $\mathrm{End}(S_2)$.

The corresponding ideals of $\mathrm{Cl}_q(\mathfrak{sl}_2)$ are given by

$$I_1 := \mathrm{Span}\left(\gamma - ct,\ v_2(\gamma - ct),\ v_{-2}(\gamma - ct),\ v_2 v_{-2}(\gamma - ct)\right),$$
$$I_2 := \mathrm{Span}\left(\gamma + ct,\ v_2(\gamma + ct),\ v_{-2}(\gamma + ct),\ v_2 v_{-2}(\gamma + ct)\right).$$

So we see that our algebra is isomorphic to $\mathrm{End}(S_1) \oplus \mathrm{End}(S_2)$. Therefore, we proved the following theorem.

Theorem

The algebra $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is isomorphic to the classical Clifford algebra $\mathrm{Cl}(\mathfrak{sl}_2)$.

$\mathrm{Cl}(\mathfrak{sl}_2)$ is generated by e, h, and f

$$e^2 = 0, \quad f^2 = 0, \quad h^2 = 2;$$
$$ef = -fe + 2, \quad eh = -he, \quad fh = -hf.$$

$\phi\colon\ \mathrm{Cl}_q(\mathfrak{sl}_2) \to \mathrm{Cl}(\mathfrak{sl}_2)$

$$\phi(v_2) = te, \qquad \phi(v_0) = \frac{\sqrt{2}}{2} t\, h \left(1 - \frac{q^2 - 1}{2q^2} ef\right), \qquad \phi(v_{-2}) = \frac{t}{2q} f,$$

Definition

The q-*deformed noncommutative Weil algebra of* $\mathfrak{sl}_2$ is a super algebra

$$\mathcal{W}_q(\mathfrak{sl}_2) := U_q(\mathfrak{sl}_2) \underline{\otimes} \mathrm{Cl}_q(\mathfrak{sl}_2).$$

with the associative multiplication given by

$$(x \otimes v) \cdot (y \otimes w) = \sum_i xy_i \otimes v_i w,$$

where

$$\sigma_R(v \otimes y) = \sum_i y_i \otimes v_i$$

and $x, y \in U_q(\mathfrak{sl}_2)$, $v, w \in \mathrm{Cl}_q(\mathfrak{sl}_2)$.

Clearly, $\mathcal{W}_q(\mathfrak{sl}_2)$ is an associative algebra in the braided monoidal category of $U_q(\mathfrak{sl}_2)$-modules with the braiding given by the universal R-matrix.

Set

$$X := v_2 = E, \quad Z := v_0 = q^{-2}EF - FE, \quad Y := v_{-2} = KF,$$

$$C := EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2}, \qquad \text{(quantum Casimir)}$$

$$W := K^{-1}.$$

Note that the elements X, Z, Y, C, and W generate $U_q(\mathfrak{sl}_2)$. Consider the following element of $U_q(\mathfrak{sl}_2)\underline{\otimes}\,\mathrm{Cl}_q(\mathfrak{sl}_2)$

$$D := \frac{1}{c}\left(X \otimes v_{-2} + \frac{q}{1+q^2} Z \otimes v_0 + q^{-2} Y \otimes v_2\right) - \frac{(q^2-1)^2}{2q(q^2+1)c^2} C \otimes \underbrace{(v_2 v_0 v_{-2} + c v_0)}_{\gamma}.$$

Theorem

$$D^2 = \frac{(q^2+1)(q^2-1)^2}{4q^3 c} C^2 \otimes 1 - \frac{q(q^2+1)}{(q^2-1)^2 c} 1 \otimes 1.$$

So D^2 is a central element in $\mathcal{W}_q(\mathfrak{sl}_2)$.

Let

$$C_q = 2FE + \frac{2q^3K + 2qK^{-1} - 1 - q^2}{(q^2-1)^2} = 2C - 2\frac{q^2+1}{(q^2-1)^2}.$$

Note that

$$\lim_{q\to 1} C_q = \mathrm{Cas}_{\mathfrak{sl}_2} = ef + fe + \tfrac{1}{2}h^2.$$

Then

$$D = \frac{1}{c}\left(X \otimes v_{-2} + \frac{q}{1+q^2} Z \otimes v_0 + q^{-2} Y \otimes v_2\right)$$
$$- \left(\frac{(q^2-1)^2}{4q(q^2+1)c^2} C_q + \frac{1}{2qc^2}\right) \otimes \left(v_2 v_0 v_{-2} + c v_0\right).$$

$$D^2 = \frac{(1+q^2)(q^2-1)^2)}{16q^3c} C_q^2 \otimes 1 + \frac{(q^2+1)^2}{4q^3c} C_q \otimes 1 + \frac{q^2+1}{4qc} 1 \otimes 1.$$

If $|\lim_{q\to 1} \frac{1}{c}| < \infty$, then

$$\lim_{q\to 1} D^2 = \left(\lim_{q\to 1} \frac{1}{c}\right)\left(\mathrm{Cas}_{\mathfrak{sl}_2} + \frac{1}{2}\right).$$

Note that $\mathrm{tr}(\mathrm{Cas}_{\mathfrak{sl}_2}) = 12$ and $D_{\mathfrak{sl}_2} = \mathrm{Cas}_{\mathfrak{sl}_2} + \frac{1}{2}$.

Let $\lambda \in \mathbb{C}$. Recall that the type I Verma $U_q(\mathfrak{sl}_2)$-module $M_{\lambda\pi}$ with the highest weight $\lambda\pi$ is defined to be an infinite-dimensional vector space

$$M_{\lambda\pi} := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C} v_{\lambda-2m}$$

equipped with the action

$$Ev_{\lambda-2m} = [\lambda - m + 1]_q v_{\lambda-2(m-1)}, \qquad Fv_{\lambda-2m} = [m+1]_q v_{\lambda-2(m+1)},$$
$$K^{\pm 1} v_{\lambda-2m} = q^{\pm(\lambda-2m)} v_{\lambda-2m},$$

where

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

If $\lambda \in \mathbb{Z}_+$, then $M_{\lambda\pi}$ has the simple $(\lambda+1)$-dimensional sub-quotient $V_{\lambda\pi}$ which is spanned by $w_{\lambda-2k}$ for $k = 0, \ldots, \lambda$. The formulas for the $U_q(\mathfrak{sl}_2)$-action on $V_{\lambda\pi}$ stay the same assuming that $w_{-\lambda-2} = 0$.

Let

$$f_A\colon A\otimes V\to V \quad \text{and} \quad f_B\colon B\otimes W\to W$$

be structure maps of an A-action, resp. B-action, on V, resp. W, then the structure map $f_{A\underline{\otimes}B}$ of $A\underline{\otimes}B$-action on $V\otimes W$ is given by

$$f_{A\underline{\otimes}B}=(f_A\otimes f_B)\circ(\mathrm{id}_A\otimes\sigma_R\otimes\mathrm{id}_W)\colon A\otimes B\otimes V\otimes W\to V\otimes W.$$

In what follows we use this formula to define an action of $\mathcal{W}_q(\mathfrak{sl}_2)$ on $M_{\lambda\pi}\otimes S_i$ and $V_{\lambda\pi}\otimes S_i$ for $i=1,2$.

Let S be one of two spin modules of $\mathrm{Cl}_q(\mathfrak{sl}_2)$. Note that

$$C = \frac{q}{(q^2-1)^2}(q^2 K + K^{-1}) + FE.$$

Therefore, the Casimir C acts on $M_{\lambda\pi}$ as

$$\frac{q}{(q^2-1)^2}(q^{2+\lambda} + q^{-\lambda}) \,\mathrm{id}\,.$$

Thus, D^2 acts on $M_{\lambda\pi} \otimes S$ as

$$\frac{q^2+1}{4qc}(q^{2+\lambda} - q^{-\lambda}) \,\mathrm{id}\,.$$

Which is nonzero if $\lambda \neq -1$.
Let M be an $U_q(\mathfrak{sl}_2)$-module, then $D \in \mathcal{W}_q(\mathfrak{sl}_2)$ acts on $M \otimes S$.
We define *the Dirac cohomology of* M to be the vector space

$$H_D(M) = \ker(D)/(\mathrm{im}(D) \cap \ker(D)).$$

Lemma

Let $\lambda \in \mathbb{C} \setminus \{-1\}$ *and* $k \in \mathbb{Z}_+$, *then* $H_D(M_{\lambda\pi}) = H_D(V_{k\pi}) = 0$.

Let $\lambda \neq -1$. The eigenvalues of D on $M_{\lambda\pi} \otimes S_1$ are

$$-\tfrac{1}{2c}[\lambda+1]_q t, \qquad \tfrac{1}{2c}[\lambda+1]_q t.$$

For $\lambda \notin \mathbb{Z}_{\geq 0}$, eigenvectors of D corresponding to the eigenvalue $-\frac{1}{2c}[\lambda+1]_q t$ are

$$\frac{q^{1-k+\lambda}(q^{2k}-1)}{q^{2k}-q^{2\lambda+2}} w_{\lambda-2k} \otimes s_1 + w_{\lambda-2(k-1)} \otimes s_{-1} \qquad \text{for } k = 1, 2, \ldots,$$

eigenvectors of D corresponding to the eigenvalue $\frac{1}{2c}[\lambda+1]_q t$ are

$$w_\lambda \otimes s_1, \qquad q^{1-k+\lambda} w_{\lambda-2k} \otimes s_1 + w_{\lambda-2(k-1)} \otimes s_{-1} \qquad \text{for } k = 1, 2, \ldots.$$

Let $\lambda \in \mathbb{Z}_{\geq 0}$. The eigenvalues of D on $V_{\lambda\pi} \otimes S_1$ are the same as for $M_{\lambda\pi} \otimes S_1$. The eigenvector of D on $V_{\lambda\pi} \otimes S_1$ corresponding to the eigenvalue $-\frac{1}{2c}[\lambda+1]_q t$ are

$$w_{-\lambda-2} \otimes s_1,$$

$$\frac{q^{1-k+\lambda}(q^{2k}-1)}{q^{2k}-q^{2\lambda+2}} w_{\lambda-2k} \otimes s_1 + w_{\lambda-2(k-1)} \otimes s_{-1} \quad \text{for } k = 1, \ldots, \lambda.$$

The eigenvector of D on $V_{\lambda\pi} \otimes S_1$ corresponding to the eigenvalue $\frac{1}{2c}[\lambda+1]_q t$ are

$$w_\lambda \otimes s_1, \quad q^{1-k+\lambda} w_{\lambda-2k} \otimes s_1 + w_{\lambda-2(k-1)} \otimes s_{-1} \quad \text{for } k = 1, \ldots, \lambda.$$

Let G be a compact Lie group and $\mathfrak{g}$ be its Lie algebra.
Let $\Lambda[\xi]$ be the Grassmann algebra with generator ξ.
$\mathrm{d} := \partial_\xi \in \operatorname{Der}\Lambda[\xi]$
Set

$$\widehat{\mathfrak{g}} := \widehat{\mathfrak{g}}_{-1} \oplus \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_{-1} = \mathfrak{g} \otimes \Lambda[\xi] \inplus \mathbb{C}\mathrm{d}.$$

For $x \in \mathfrak{g}$, let $L_x = x \otimes 1 \in \widehat{\mathfrak{g}}_0$, $\iota_x = x \otimes \xi \in \widehat{\mathfrak{g}}_{-1}$.
The non-zero brackets are

$$[L_x, L_y] = L_{[x,y]}, \quad [L_x, \iota_y] = \iota_{[x,y]}, \quad [\iota_x, \mathrm{d}] = L_x \qquad \text{for } x, y \in \mathfrak{g}.$$

A $\mathfrak{g}$-*differential spaces* is a superspace B, together with a $\widehat{\mathfrak{g}}$-modules structure $\rho\colon \widehat{\mathfrak{g}} \to \operatorname{End}(B)$.
A $\mathfrak{g}$-*differential algebra* is a superalgebra B, equipped with a structure of G-differential space such that $\rho(x) \in \operatorname{Der} B$ for all $x \in \widehat{\mathfrak{g}}$.

Take $B = \Lambda\mathfrak{g}^*$, equipped with the coadjoint action of $\mathfrak{g}$.

- e_i be a basis in $\mathfrak{g}$ and f_i be the dual basis in $\mathfrak{g}^* \simeq \Lambda^1\mathfrak{g}^*$

$$[e_i, e_j] = \sum_k c_{i,j}^k e_k$$

- The contractions ι_{e_i} are defined by

$$\iota_{e_i} f_j = \langle f_j, e_i \rangle, \qquad \iota_{e_i}(x \wedge y) = (\iota_{e_i} x) \wedge y + (-1)^{\deg x} x \wedge \iota_{e_i} y.$$

- The Lie derivatives are given by

$$L_{e_i} = -\sum_{k,j} c_{i,j}^k \, f_j \wedge \iota_{e_k}.$$

- The differential d is given by Koszul's formula

$$\mathrm{d}_\wedge = \frac{1}{2} \sum_a f_a \wedge L_{e_a}.$$

Then $\Lambda\mathfrak{g}^*$ is a $\mathfrak{g}$-differential algebra.
One can show that $H(\Lambda\mathfrak{g}^*, \mathrm{d}) \cong (\Lambda\mathfrak{g}^*)^G \cong H(\mathfrak{g})$.

Suppose

- $\mathfrak{g}$ has an nondegenerate invariant symmetric bilinear form.
- e_a be an orthonormal basis of $\mathfrak{g}$,

$$[e_a, e_b] = \sum_k c_{ab}^k e_k.$$

- Using the invariant bilinear form on $\mathfrak{g}$ to pull down indices

$$f_{ab}^c \longmapsto f_{abc}$$

Set

$$g_a = -\frac{1}{2}\sum_{r,s} f_{ars} e_r e_s \in \mathrm{Cl}^{(2)}(\mathfrak{g}),$$

$$\gamma = \frac{1}{3}\sum_a e_a g_a = -\frac{1}{6}\sum_{a,b,c} f_{abc} e_a e_b e_c \in \mathrm{Cl}^{(3)}(\mathfrak{g})^{\mathfrak{g}}.$$

The Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ with derivations

$$\iota_a = [e_a, \cdot\,]_{\mathrm{Cl}}, \quad L_a = [g_a, \cdot\,]_{\mathrm{Cl}}, \quad \mathrm{d}_{\mathrm{Cl}} = [\gamma, \cdot\,]_{\mathrm{Cl}}.$$

The cohomology is trivial in all filtration degrees (except if $\mathfrak{g}$ is abelian, in which case $\mathrm{d}_{\mathrm{Cl}} = 0$).

Set

$$[x,y]_\sigma := \left(m_{\mathrm{Cl}_q} - (-1)^{p(x)p(y)} m_{\mathrm{Cl}_q} \circ \sigma\right)(x \otimes y) \quad \text{for } x, y \in \mathrm{Cl}_q(\mathfrak{sl}_2),$$

where m_{Cl_q} denotes the multiplication map in $\mathrm{Cl}_q(\mathfrak{sl}_2)$.

$$\mathrm{d}_{\mathrm{Cl}}(x) = \gamma x - (-1)^{p(x)} x\gamma = [\gamma, x]_\sigma.$$

For all $x \in \mathrm{Cl}(\mathfrak{g})$ set

$$L_{v_k}(x) = \mathrm{ad}_{v_k} x \qquad \text{for } k = 2, 0, -2.$$

Lemma

We have that $L_{v_k}(x) = [-\mathrm{d}_{\mathrm{Cl}}(v_k), x]_\sigma$.

For $x \in \mathrm{Cl}_q(\mathfrak{sl}_2)$ define

$$\iota_{v_2}(x) := [-v_2, x]_\sigma, \quad \iota_{v_0}(x) := [-v_0, x]_\sigma, \quad \iota_{v_0}(x) := [-v_{-2}, x]_\sigma.$$

Lemma

We have that $L_v = [\iota_v, \mathrm{d}_{\mathrm{Cl}_q}]_\sigma$ *for* $v \in V_{2\pi}$.

A q-deformed $\mathfrak{sl}_2$-differential algebras is an algebra B together with
1) an action of $U_q(\mathfrak{sl}_2)$, in particular, we can define a Lie derivative L_x with respect an element $x \in V_{2\pi} \subset U_q(\mathfrak{sl}_2)$,
2) an action ι_x of $\Lambda_q V_{2\pi}$,
3) an differential d,
such that $L_x = [\iota_x, \mathrm{d}]$ for all $x \in V_{2\pi}$.

Theorem

The algebra $\mathrm{Cl}_q(\mathfrak{sl}_2)$ *admits a structure of* q*-deformed* $\mathfrak{sl}_2$*-differential algebra.*

Let $\Lambda(n)$ denotes the Grassmann algebra with n generators $\xi_1, \ldots, \xi_n$. The Grassmann algebra $\Lambda(n)$ has a natural $\mathbb{Z}$-grading given by $\deg \xi_i = 1$. Let $\mathfrak{vect}(0|n) := \mathrm{Der}\,\Lambda(n)$. Clearly, $\mathfrak{vect}(0|n)$ is a $\mathbb{Z}$-graded Lie superalgebra where $\deg \partial_{\xi_i} = -1$. Let $\mathfrak{vect}(0|n)_{-1}$ denotes the homogeneous component of degree -1.

Any semisimple Lie superalgebra is the direct sum of the following summands

$$\tilde{\mathfrak{s}} \otimes \Lambda(n) \ni\!\!\!\!+ \mathfrak{v},$$

where $\mathfrak{s}$ is a simple Lie superalgebra and $\mathfrak{v} \subset \mathfrak{vect}(0|n)$ such that $\mathfrak{s} \subseteq \tilde{\mathfrak{s}} \subseteq \mathrm{Der}\,\mathfrak{s}$ and the projection $\mathfrak{v} \to \mathfrak{vect}(0|n)_{-1}$ is onto.

In our case (for $\widehat{\mathfrak{g}}$) we have that $n = 1$, $\mathfrak{v} = \mathrm{Span}_{\mathbb{C}}(\partial_\xi)$, $\tilde{\mathfrak{s}} = \mathfrak{s} = \mathfrak{g}$.

$\mathfrak{sl}_2 \otimes \Lambda(1)$ “q-deforms” to $U_q(\mathfrak{sl}_2) \otimes \Lambda_q V_{2\pi}$.

Thank you